

$$C_{1111}^m = c_{111}^m + 2c_{112}^m$$

$$= -\left[3B^T\left(\frac{\partial C_{11}^s}{\partial P}\right)_T + 3B^T + c_{11}^s\right] = C_a^m \quad (36)$$

$$C_{1122}^m = 2c_{112}^m + c_{123}^m$$

$$= -\left[3B^T\left(\frac{\partial C_{12}^s}{\partial P}\right)_T - 3B^T + c_{12}^s\right] = C_b^m \quad (37)$$

$$C_{1212}^m = c_{144}^m + 2c_{166}^m$$

$$= -\left[3B^T\left(\frac{\partial C_{44}^s}{\partial P}\right)_T + 3B^T + c_{44}^s\right] = C_c^m \quad (38)$$

Alternately, the expressions for C_a^m , C_b^m and C_c^m can be obtained directly from the effective elastic constants given earlier by taking the pressure derivatives and evaluating them at zero-pressure. Thus, we find

$$C_a^m = -3B^T\left[1 + \left\{\frac{\partial C_{11}^s}{\partial P}\right\}_{P=0}\right] - c_{11}^s \quad (36a)$$

$$C_b^m = -3B^T\left[-1 + \left\{\frac{\partial C_{12}^s}{\partial P}\right\}_{P=0}\right] - c_{12}^s \quad (37a)$$

$$C_c^m = -3B^T\left[1 + \left\{\frac{\partial C_{44}^s}{\partial P}\right\}_{P=0}\right] - c_{44}^s \quad (38a)$$

To obtain the relation of the second pressure derivatives of the elastic constants to partial contractions of the higher-order elastic constants, we start from equation (34). By taking the pressure derivatives of $(\partial C_{ijkl}^s / \partial P)_T$ and arranging the result,

$$\left(\frac{\partial^2 C_{ijkl}^s}{\partial P^2}\right)_T = \frac{1}{(3B^T)^2} \left[C_{ijkl}^s \left\{1 + 3\left(\frac{\partial B^T}{\partial P}\right)_T\right\} \right. \\ \left. + \left\{4 + 3\left(\frac{\partial B^T}{\partial P}\right)_T\right\} C_{ijklmm}^m + C_{ijklmnn}^m \right] \quad (39)$$

where $C_{ijklmnn}^m$ are certain linear combinations of the fourth-order elastic constants and

$$\left(\frac{\partial B^T}{\partial P}\right)_T = -\frac{1}{9B^T} [C_a^T + 2C_b^T] \quad (40)$$

Therefore, solving equation (39) for $C_{ijklmnn}^m$, we find explicitly the followings:

$$C_a^m + 2C_e^m = (6B^T - C_a^T - 2C_b^T) \left[1 + \left(\frac{\partial C_{11}^s}{\partial P}\right)_T\right] \\ + c_{11}^s - 2C_a^m + (3B^T)^2 \left(\frac{\partial^2 C_{11}^s}{\partial P^2}\right)_T = C_A^m \quad (41)$$

$$2C_e^m + 3C_{1123}^m = (6B^T - C_a^T - 2C_b^T) \left[-1 + \left(\frac{\partial C_{12}^s}{\partial P}\right)_T\right] \\ + c_{12}^s - 2C_b^m + (3B^T)^2 \left(\frac{\partial^2 C_{12}^s}{\partial P^2}\right)_T = C_B^m \quad (42)$$

$$C_f^m + 2C_g^m = (6B^T - C_a^T - 2C_b^T) \left[1 + \left(\frac{\partial C_{44}^s}{\partial P}\right)_T\right] \\ + c_{44} - 2C_c^m + (3B^T)^2 \left(\frac{\partial^2 C_{44}^s}{\partial P^2}\right)_T = C_C^m \quad (43)$$

Or, from the effective elastic constants given by equations (22-24), we find that

$$C_A^m = c_{11}^s - 2C_a^m + (6B^T - C_a^T - 2C_b^T) \\ \times \left[1 + \left\{\frac{\partial C_{11}^s}{\partial P}\right\}_{P=0}\right] \\ + (3B^T)^2 \left\{\frac{\partial^2 C_{11}^s}{\partial P^2}\right\}_{P=0} \quad (41a)$$

$$C_B^m = c_{12}^s - 2C_b^m + (6B^T - C_a^T - 2C_b^T) \\ \times \left[-1 + \left\{\frac{\partial C_{12}^s}{\partial P}\right\}_{P=0}\right] \\ + (3B^T)^2 \left\{\frac{\partial^2 C_{12}^s}{\partial P^2}\right\}_{P=0} \quad (42a)$$

$$C_C^m = c_{44} - 2C_c^m + (6B^T - C_a^T - 2C_b^T) \\ \times \left[1 + \left\{\frac{\partial C_{44}^s}{\partial P}\right\}_{P=0}\right] \\ + (3B^T)^2 \left\{\frac{\partial^2 C_{44}^s}{\partial P^2}\right\}_{P=0} \quad (43a)$$

Thus, from equations (28-31) and equations (41-43), we have

$$C_A^m = c_{1111}^m + 4c_{1112}^m + 2c_{1122}^m + 2c_{1123}^m \quad (44)$$

$$C_B^m = 2c_{112}^m + 2c_{122}^m + 5c_{123}^m \quad (45)$$

$$C_C^m = c_{1144}^m + 2c_{1155}^m + 4c_{1255}^m + 2c_{1266}^m \quad (46)$$

These are the primary experimental quantities that are resulting from the ultrasonic-pressure experiments at high pressures.

5. SUMMARY AND CONCLUDING NOTES

Summarizing the foregone sections, the followings may be stated:

(i). Expressions for the effective elastic constants of a cubic crystal subjected to moderately high hydrostatic pressure are derived from the consideration of the rigorous stress-strain relation set by Murnaghan's theory of finite deformations, and these have been compared with the earlier work. Discrepancies were found in the expressions of the effective second-order elastic constants C_{12} and C_{44} . The expressions as ones presented here may be derived from the strain-energy density considerations, and this method may be used to distinguish the observed discrepancies.

(ii). Expressions for the effective second-order elastic constants are

$$C_{\mu\nu} = c_{\mu\nu} + C_I\eta + C_{II}\eta^2$$

where C_I and C_{II} are certain linear combinations of the second- and the higher-order elastic constants of crystal and η is the Lagrangian strain which depends upon pressure. The present expressions are distinguished from the expressions of Birch[1], Seeger and Buck[4], and Thurston[5] and others mainly by the appearance of the $C_{II}\eta^2$ terms in the expressions of the effective elastic constants of cubic crystals.

(iii). Ultrasonic effective second-order elastic constants at high hydrostatic pressures

were derived and presented in terms of *thermodynamically mixed* higher-order elastic constants. The rigorous relationships between the pressure derivatives of the elastic constants and partial contractions of the higher-order elastic constants were presented also. And, then, the primary experimental quantities that may be resulting from the ultrasonic-pressure experiments have been identified in a useful form in terms of the *thermodynamically mixed* second-, third-, and fourth-order elastic constants of the crystal under the study.

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